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DOUBLE PILING STRUCTURE OF MATRIX MONOTONE FUNCTIONS AND OF MATRIX CONVEX FUNCTIONS II

HIROYUKI OSAKA[A] AND JUN TOMIYAMA

Abstract. We continue the analysis in [H. Osaka and J. Tomiyama, Double piling structure of matrix monotone functions and of matrix convex functions, Linear and its Applications **431**(2009), 1825 - 1832] in which the followings three assertions at each label n are discussed:

(1) $f(0) \le 0$ and f is n-convex in $[0, \alpha)$

(2) For each matrix a with its spectrum in $[0, \alpha)$ and a contraction c in the matrix algebra M_n,

$$f(c^*ac) \le c^*f(a)c.$$

(3) The function $f(t)/t$ $(= g(t))$ is n-monotone in $(0, \alpha)$.

We know that two conditions (2) and (3) are equivalent and if f with $f(0) \le 0$ is n-convex, then g is $(n-1)$-monotone. In this note we consider several extra conditions on g to conclude that the implication from (3) to (1) is true. In particular, we study a class $Q_n([0, \alpha))$ of functions with conditional positive Lowner matrix which contains the class of matrix n-monotone functions and show that if $f \in Q_{n+1}([0, \alpha))$ with $f(0) = 0$ and g is n-monotone, then f is n-convex. We also discuss about the local property of n-convexity.

1. Introduction

Let $n \in \mathbf{N}$ and M_n be the algebra of $n \times n$ matrices. We call a function f matrix convex of order n or n-convex in short whenever the inequality

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda)f(B),\ \ \lambda \in [0,1]$$

holds for every pair of selfadjoint matrices $A, B \in M_n$ such that all eigenvalues of A and B are contained in I. Matrix monotone functions on I are similarly defined as the inequality

$$A \le B \Longrightarrow f(A) \le f(B)$$

for an arbitrary selfadjoint matrices $A, B \in M_n$ such that $A \le B$ and all eigenvalues of A and B are contained in I.

We denote the spaces of operator monotone functions and of operator convex functions by $P_\infty(I)$ and $K_\infty(I)$ respectively. The spaces for n-monotone functions and n-convex functions are written as $P_n(I)$ and $K_n(I)$. We have then

$$P_1(I) \supseteq \cdots \supseteq P_{n-1}(I) \supseteq P_n(I) \supseteq P_{n+1}(I) \supseteq \cdots \supseteq P_\infty(I)$$
$$K_1(I) \supseteq \cdots \supseteq K_{n-1}(I) \supseteq K_n(I) \supseteq K_{n+1}(I) \supseteq \cdots \supseteq K_\infty(I)$$

Here we meet the facts that $\cap_{n=1}^\infty P_n(I) = P_\infty(I)$ and $\cap_{n=1}^\infty K_n(I) = K_\infty(I)$. We regard these two decreasing sequences as noncommutative counterpart of the classical piling sequence $\{C^n(I), C^\infty(I), Anal(I)\}$, where the class $Anal(I)$ denotes the set of all of analytic functions over I. We could understand that the class of operator monotone functions $P_\infty(I)$ corresponds to the class $\{C^\infty(I), Anal(I)\}$ by the famous characterization of those functions by Loewner as the restriction of Pick functions.

In these circumstances, it will be well recognized that we should not stick our discussions only to those classes $P_\infty(I)$ and $K_\infty(I)$, that is, the class of operator monotone functions and that of operator convex functions. Those classes $\{P_n(I)\}$ and $\{K_n(I)\}$ are not merely optional ones to $P_\infty(I)$ and $K_\infty(I)$. They should play important roles in the aspect of noncommutative calculus as the ones $\{C^n(I)\}$ play in usual (commutative) calculus.

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The first basic question is whether $P_{n+1}(I)$ (resp. $K_{n+1}(I)$) is strictly contained in $P_n(I)$ (resp. $K_n(I)$) for every n. This gap problem for arbitrary n has been solved only recently ([7], [16], [10]).

On the other hand, there are basic equivalent assertions known only at the level of operator monotone functions and operator convex functions by [8], [9]. We shall discuss those (equivalent) assertions as the correlation problem between two kinds of piling structures $\{P_n(I)\}$ and $\{K_n(I)\}$, that is, we are planning to discuss relations between those assertions at each level n.

In [17] we discussed about the following 3 assertions at each level n among them in order to see clear insight of the aspect of the problems:

(i) $f(0) \leq 0$ and f is n-convex in $[0, \alpha)$,

(ii) For each matrix a with its spectrum in $[0, \alpha)$ and a contraction c in the matrix algebra M_n,

$$f(c^\star ac) \leq c^\star f(a)c,$$

(iii) The function $\frac{f(t)}{t}$ $(= g(t))$ is n-monotone in $(0, \alpha)$.

Then we showed that for each n the condition (ii) is equivalent to the condition (iii) and the assertion that f is n-convex with $f(0) \leq 0$ implies that $g(t)$ is $(n-1)$-monotone holds.

In this note we continue to consider the double piling structure in [17] and focus our discussion to the class $Q_n(I)$ of all real C^1 functions f on the interval I such that for each $\lambda_1, \lambda_2, \ldots, \lambda_n \in I$ the corresponding Lowner matrix $([\lambda_i, \lambda_j]_f)$ is an almost positive matrix ([5, IV]). We discuss about the relation between the condition (i) and the condition (iii). For example, we show that if f is 2-convex on $[0, \alpha)$, then $g(t)(= \frac{f(t)}{t})$ belongs to $Q_2(0, \alpha)$. Conversely, when $f \in Q_n([0, \alpha))$ with $f(0) = 0$, if $g(t)$ is (n - 1)-monotone, then f is $(n-1)$-convex. We note that for each $n \in \mathbf{N}$ the class $P_n(I)$ is a subset of $Q_n(I)$ and $K_n(I) \cap Q_n(I) \neq \emptyset$. It could be that $P_n(I)$ is a proper subset of $Q_n(I)$. In fact we know that $t^2 \in Q_2(0, \alpha) \backslash P_2(0, \alpha)$ for any $\alpha > 0$. On the contrary, for $0 < \alpha < 1$, $n \geq 3$ and $\beta > 0$, then $t^\alpha \in Q_n([0, \beta))$, but $t^\alpha \notin K_{n-1}([0, \beta))$.

The authors are indebted to a recent work [12] by F. Hiai and T. Sano for giving an attension to the class $Q_n(I)$.

2. Preliminary

We shall sometimes use the standard regularization procedure, cf. for example Donoghu [5, p11]. Let ϕ be a positive and even C^∞-function defined on the real axis, vanishing ooutside the closed interval $[-1, 1]$ and normalized such that

$$\int_{-1}^{1} \phi(x) = 1.$$

For any locally intergrable function f defined in an open interval (a, b) we form its regularization

$$f_\varepsilon(t) = \frac{1}{\varepsilon} \int_a^b \phi(\frac{t-s}{\varepsilon}) f(s) ds, \quad t \in \mathbf{R}$$

for small $\varepsilon > 0$, and realize that it is infinitely many times differentiable. For $t \in (a + \varepsilon, b - \varepsilon)$ we may also write

$$f_\varepsilon(t) = \int_{-1}^{1} \phi(s) f(t - \varepsilon s) ds.$$

If f is continuous, then f_ε converges uniformly to f on any compact subinterval of (a, b). If in addition f is n-convex (or n-monotone) in (a, b), then f_ε is n-convex (or n-monotone) in the slightly smaller interval $(a + \varepsilon, b - \varepsilon)$. Since the pointwise limit of a sequence of n-convex (or n-monotone) functions is again n-convex (or n-monotone), we may therefore in many applications assume that an n-convex or n-monotone function is sufficiently many times differentiable.

For a sufficiently smooth function $f(t)$ we denote its n-th divided difference for n-tuple of points $\{t_1, t_2, \ldots, t_n\}$ defined as, when they are all different,

$$[t_1, t_2]_f = \frac{f(t_1) - f(t_2)}{t_1 - t_2}, \text{and inductively}$$

$$[t_1, t_2, \ldots, t_n]_f = \frac{[t_1, t_2, \ldots, t_{n-1}]_f - [t_2, t_3, \ldots, t_n]_f}{t_1 - t_n}.$$

And when some of them coincides such as $t_1 = t_2$ and so on, we put as

$$[t_1, t_1]_f = f'(t_1) \quad \text{and inductively}$$

$$[t_1, t_1, \ldots, t_1]_f = \frac{f^{(n)}(t_1)}{n!}.$$

When there appears no confusion we often skip the referring function f. We notice here the most important property of divided differences is that it is free from permutations of $\{t_1, t_2, \ldots, t_n\}$ in an open interval I.

Proposition 2.1. (1) (Ia) Monotonicity(Loewner 1934 [15])

$$f \in P_n(I) \Longleftrightarrow ([t_i, t_j]) \geq 0 \text{ for any } \{t_1, t_2, \ldots, t_n\}$$

(IIa) Convexity (Kraus 1936 [14])

$$f \in K_n(I) \Longleftrightarrow ([t_1, t_i, t_j]) \geq 0 \text{ for any } \{t_1, t_2, \ldots, t_n\},$$

where t_1 can be replaced by any (fixed) t_k.

(2) (Ib) Monotonicity (Loewner 1934 [15], Dobsch 1937 [4]-Donoghue 1974[5])
For $f \in C^{2n-1}(I)$

$$f \in P_n(I) \Longleftrightarrow M_n(f;t) = \left(\frac{f^{(i+j-1)}(t)}{(i+j-1)!}\right) \geq 0 \ \forall t \in I$$

(IIb) Convexity (Hansen-Tomiyama 2007[10]) For $f \in C^{2n}(I)$

$$f \in K_n(I) \Longrightarrow K_n(f;t) = \left(\frac{f^{(i+j)}(t)}{(i+j)!}\right) \geq 0 \ \forall t \in I.$$

In particular, for $n = 2$ the converse is also true.

We remind that to prove the implication $M_n(f;t) \geq 0 \Rightarrow f \in P_n(I)$ in (Ib) the local property for the monotonicity plays an essential role. Similarily to prove the converse implication in the criterion of convexity in (IIb) in the above proposition we need **the local property conjecture for the convexity**, that is, if f is n-convex in the intervals (a, b) and (c, d) $(a < c < b < d)$, then f is n-convex on (a, d).

Now we have only a partial sufficiency, that is, if $K_n(f;t_0)$ is positive, then there exists a neighborhood of t_0 on which f is n-convex. (See [10, Theorem 1.2] for example.)

Though the method for the implication $(II_b) \Rightarrow (IIa)$ under the assumption of the local property theorem for the convexity may be familiar for some specialist, we provide here the proof for readers' convenience.

Proposition 2.2. Let $f \in C^{2n}(I)$ such that $K_n(f;t) = \left(\frac{f^{(i+j)}(t)}{(i+j)!}\right) \geq 0 \ \forall t \in I$. Suppose that n-convexity has the local property. Then $f \in K_n(I)$.

Proof. If $K_n(f;t) > 0$ for $t \in I$, for each $t \in [0, \alpha)$ there is an open interval I_t such that f is n-convex in I_t (c.f. [10, Theorem 1.2]). Hence for any compact subset K in I there are finitely many intervals $I_{t_i} \subset I$ such that $K \subset \cup_{i=1}^{l} I_{t_i}$. From the local property for n-convexity, we conclude that f is n-convex on K. Taking a sequence of closed intervals K_l such that $K_l \subset K_{l+1}$ and $I = \cup_{l=1}^{\infty} K_l$. Since the restriction of f to K_l is n-convex for each l, we know that f is n-convex on I.

In the case that $K_n(f;t)$ is non-negative, we choose a function h for which $K_n(h;t) > 0$ as in [11, Proposition 2.1]. Then for each positive number $\varepsilon > 0$ we have

$$K_n(f + \varepsilon h; t) = K_n(f;t) + \varepsilon K_n(h;t) > 0$$

for all $t \in I$, and we know that $f + \varepsilon h$ is n-convex from the first observation. Hence we can conclude that f is n-convex. □

3. The class Q_n

In this secton we introduce the class $Q_n(I)$ on an interval I and its characterization from [5].

Definition 3.1. Let $H^n = \{x = (x_1, \dots, x_n) \in \mathbf{C}^n \mid \sum_{i=1}^n x_i = 0\}$. An $n \times n$ Hermitian matrix A is said to be *conditionally positive definite* (or almost positive) if

$$(x \mid Ax) \geq 0$$

for all $x \in H^n$ and *conditionally negative definite* if $-A$ is conditionally positive definite.

Example 3.2. For $n \in \mathbf{N}$ the matrix $(i+j)_{1\leq i,j,\leq n}$ is conditional positive and conditional negative. Indeed, for any $x = (x_1, \dots, x_n) \in H^n$

$$\begin{aligned}
(x \mid (i+j)x) &= \sum_{i,j=1}^n (i+j)x_i\overline{x_j} \\
&= \sum_{i,j=1}^n ix_i\overline{x_j} + \sum_{i,j=1}^n jx_i\overline{x_j} \\
&= (\sum_{i=1}^n ix_i)\overline{\sum_{j=1}^n x_j} + (\sum_{i=1}^n x_i)\overline{\sum_{j=1}^n jx_j} \\
&= 0 + 0 = 0.
\end{aligned}$$

The following is well known but we put it for readers' convenience.

Lemma 3.3. [5, XV Lemma 1] [2, Exercise 5.6.16] Given an $n \times n$ Hermitian matrix $B = [b_{ij}]$ let D be the $(n-1) \times (n-1)$ matrix with entries

$$d_{ij} = b_{ij} + b_{i+1,j+1} - b_{i,j+1} - b_{i+1,j}.$$

Then B is conditionally positive definite if and only if D is positive semidefinite.

Let (a, b) be an interval of the real line and $n \in \mathbf{N}$ with $n \geq 2$. The calss $Q_n(a, b)$ is defined as the class of all real C^1 functions f on (a, b) such that for each $\lambda_1, \lambda_2, \dots, \lambda_n \in (a, b)$ the corresponding Lowner matrix $([\lambda_i, \lambda_j]_f)$ is an almost positive matrix. Note that $P_n(a, b) \subset Q_n(a, b)$ for each $n \in \mathbf{N}$. Since for each $n \in \mathbf{N}$ there is an example of a n-monotone and n-convex polynomial on (a, b) ([10, Proposition 1.3]), we know that $Q_n(a, b) \cap K_n(a, b) \neq \emptyset$.

The following is a characterization of the class $Q_2(a, b)$.

Lemma 3.4. [5, XV Lemma 3] A real C^1 function f belongs to $Q_2(a, b)$ if and only if the derivative f' is convex on (a, b).

The following is characterization of the class $Q_n(a, b)$.

Lemma 3.5. [5, XV Lemma 4] A real C^1 function f belongs to $Q_n(a, b)$ if and only if for every $z \in (a, b)$ the function $[x, z, z]_f$ belongs to $P_{n-1}(a, b)$.

4. $Q_2(I)$ AND THE FIRST DERIVATIVE CONDITION

We know that for an interval (a,b) $\{f' \mid f \in K_2(a,b)\} \subset Q_2(a,b)$.

The following is well-known result, but we give an elementary proof here.

Lemma 4.1. The Cauchy matrix $(\frac{1}{i+j})_{1\le i,j\le n}$ is positive definite.

Proof. For any $x = (x_1, x_2, \dots, x_n) \in \mathbf{C}^n \backslash \{(0,0,\dots,0)\}$

$$\begin{aligned}(x \mid (\frac{1}{i+j})x) &= \sum_{i,j=1}^{n} \frac{1}{i+j} x_i \overline{x_j} \\ &= \sum_{i,j=1}^{n} \int_0^1 t^{i+j-1} dt x_i \overline{x_j} \\ &= \int_0^1 t \sum_{i,j=1}^{n} t^{i+j-2} x_i \overline{x_j} dt \\ &= \int_0^1 t (\sum_{i=1}^{n} x_i t^{i-1}) \overline{(\sum_{j=1}^{n} x_j t^{j-1})} dt \\ &> 0\end{aligned}$$

□

Proposition 4.2. For an interval (a,b) we have the followings.

(1) If f is 2-convex, $f' \in Q_2(a,b)$.
(2) If f' is 2-monotone, then f is 2-convex.
(3) $e^t \in Q_2(a,b) \backslash \{f' \mid f \in K_2(a,b) \cap M_2(a,b)\}$.

Proof. (1): Let f is 2-convex. Then

$$K_2(f,t) = \begin{pmatrix} \frac{1}{2} f^{(2)}(t) & \frac{1}{6} f^{(3)}(t) \\ \frac{1}{6} f^{(3)}(t) & \frac{1}{24} f^{(4)}(t) \end{pmatrix}$$

is positive definite. Therefore both derivatives $f^{(2)}$ and $f^{(4)}$ are non-negative. Hence $(f')'$ is convex. Therefore $f' \in Q_2(a,b)$ by Lemma 3.4.

(2): For each $n \in \mathbf{N}$

$$\begin{aligned}K_n(f;t) &= (\frac{f^{(i+j)}(t)}{(i+j)!}) \\ &= (\frac{1}{i+j}) \circ (\frac{h^{(i+j-1)}(t)}{(i+j-1)!}) \\ &= (\frac{1}{i+j}) \circ M_n(h;t),\end{aligned}$$

where $\circ$ means the Hadamard product of self-adjoint matrices and $h = f'$.

Since f' is 2-monotone, $M_2(h;t) \ge 0$ for $t \in [0,\alpha)$. Since $(\frac{1}{i+j})$ is positive definite by Lemma 4.1 ([2, Exercise 1.1.2]) and the Hadmard product of positive semidfinite matrices becomes positive semidefinite, we have $K_2(f;t) \ge 0$ for $t \in [0,\alpha)$.

Hence we could conclude that f is 2-convex by [10, Theorem 2.3].

(3): Since e^t does not belong to neither $K_2(a,b)$ nor $M_2(a,b)$, $e^t \notin \{f' \mid f \in K_2(a,b) \cap M_2(a,b)\}$. Obviously $(e^t)' = e^t$ is convex, and $e^t \in Q_2(a,b)$. □

5. Double pilling structure

In this section we recall the following assertions in [17].

Let $n \in \mathbf{N}$ and $f\colon [0,\alpha) \to \mathbf{R}$ be a continuous function for some $\alpha > 0$.

$(1)_n$ f is n-convex with $f(0) \le 0$.

$(2)_n$ For an operator $A \in M_n(\mathbf{C})$ with its spectrum in $[0,\alpha)$ and a contraction C

$$f(C^*AC) \le C^* f(A) C.$$

$(3)_n$ $g(t) = \frac{f(t)}{t}$ is n-monotone on $(0,\alpha)$.

Then we know that

$$(1)_{n+1} \prec (2)_n \sim (3)_n \prec (1)_{[\frac{n}{2}]}.$$

Further, in this section we consider the relation between two assertions $(1)_n$ and $(3)_n$.

Proposition 5.1. Suppose that $f(t)$ is n-convex in $[0,\alpha)$ with $f(0) \le 0$. Let $g(t) = \frac{f(t)}{t}$. Then

1. $K_n(\int g, t)$ is positive semidefinite on $[0,\alpha)$.
2. When $n = 2$, $g(t)$ belongs to the class $Q_2(0,\alpha)$.

Proof. (1):

Since

$$f^{(k)}(t) = tg^{(k)}(t) + kg^{(k-1)}(t)$$

for $1 \le k$, we have

$$t^{(k-1)} f^{(k)}(t) = (t^{(k)} g^{(k-1)}(t))'$$

for $2 \le k$.

Since f is n-convex, $K_n(f,t) = (\frac{f^{(i+j)}(t)}{(i+j)!}) \ge 0$. From the above calculation

$$\begin{aligned}(t^{i+j-1}) \circ K_n(f,t) &= (\frac{t^{i+j-1} f^{(i+j)}(t)}{(i+j)!}) \\ &= (\frac{(t^{i+j} g^{(i+j-1)}(t))'}{(i+j)!}) \ge 0.\end{aligned}$$

Hence

$$\begin{aligned}0 &\le \int_0^t (\frac{(u^{i+j} g^{(i+j-1)}(u))'}{(i+j)!}) du \\ &= (\frac{t^{i+j} g^{i+j-1}(t)}{(i+j)!}) - \lim_{s\to 0}(\frac{s^{i+j} g^{(i+j-1)}(s)}{(i+j)!}).\end{aligned}$$

Since $f(0) \le 0$ and $((-1)^{i+j})$ is positive semidefinite, we have

$$\begin{aligned}\lim_{s\to 0}(\frac{s^{i+j} g^{(i+j-1)}(s)}{(i+j)!}) &= (\frac{(-1)^{i+j-1} f(0)}{i+j}) \\ &= (-f(0))(\frac{(-1)^{i+j}}{i+j}) \\ &= -f(0)(\frac{1}{i+j}) \circ ((-1)^{i+j}) \ge 0\end{aligned}$$

by Lemma 4.1, where $\circ$ means the Hadamard product.

Therefore $(\frac{t^{i+j} g^{(i+j-1)}(t)}{(i+j)!}) \ge 0$.

Next we will check the determinant of the principal submatrix $D(k_1, k_2, \dots, k_r) = (\frac{t^{k_i+k_j} g^{k_i+k_j-1}(t)}{(k_i+k_j)!})$ of $M_n(g,t)$ for any $k_1, k_2, \dots, k_r \in \{1, 2, \dots, n\}$ and $1 \le r \le n$. Note that since $(\frac{t^{i+j} g^{(i+j-1)}(t)}{(i+j)!})$ is positive semidefinite, $\det(D(k_1, k_2, \dots, k_r)) \ge 0$ from the standard theorem in linear algebras.

$$
\begin{aligned}
\det(D(k_1,k_2,\ldots,k_r)) &= |(\frac{t^{k_i+k_j} g^{k_i+k_j-1}(t)}{(k_i+k_j)!})| \\
&= t^{2k_1}\begin{vmatrix}
\frac{g^{(2k_1-1)}(t)}{(2k_1)!} & \frac{t^{k_2-k_1} g^{(k_1+k_2-1)}(t)}{(k_1+k_2)!} & \cdots & \frac{t^{k_r-k_1} g^{k_1+k_r-1}(t)}{(k_1+k_r)!} \\
\frac{t^{k_2+k_1} g^{(k_1+k_2-1)}(t)}{(k_2+k_1)!} & \frac{t^{2k_2} g^{(2k_2-1)}(t)}{(2k_2)!} & \cdots & \frac{t^{k_2+k_r} g^{(k_2+k_r-1)}(t)}{(k_2+k_r)!} \\
\vdots & \vdots & & \vdots \\
\frac{t^{k_r+k_1} g^{(k_1+k_r-1)}(t)}{(k_1+k_r)!} & \frac{t^{k_r+k_2} g^{(k_2+k_r-1)}(t)}{(k_2+k_r)!} & \cdots & \frac{t^{2k_r} g^{(2k_r-1)}(t)}{(2k_r)!}
\end{vmatrix} \\
&= t^{2k_1} t^{k_1+k_2}\begin{vmatrix}
\frac{g^{(2k_1-1)}(t)}{(2k_1)!} & \frac{t^{k_2-k_1} g^{(k_1+k_2-1)}(t)}{(k_1+k_2)!} & \cdots & \frac{t^{k_r-k_1} g^{(k_1+k_r-1)}(t)}{(k_1+k_r)!} \\
\frac{g^{(k_2+k_1-1)}(t)}{(k_1+k_2)!} & \frac{t^{k_2-k_1} g^{(2k_2-1)}(t)}{(2k_2)!} & & \frac{t^{k_r-k_1} g^{(k_2+k_r-1)}(t)}{(k_2+k_r)!} \\
\vdots & \vdots & & \vdots \\
\frac{t^{k_r+k_1} g^{(k_r+k_1-1)}(t)}{(k_r+k_1)!} & \frac{t^{k_r+k_2} g^{(k_r+k_2-1)}(t)}{(k_r+k_2)!} & \cdots & \frac{t^{2k_r} g^{(2k_r-1)}(t)}{(2k_r)!}
\end{vmatrix} \\
&= \cdots \\
&= t^{2k_1} t^{k_2+k_1}\cdots t^{k_r+k_1}\begin{vmatrix}
\frac{g^{(2k_1-1)}(t)}{(2k_1)!} & \frac{t^{k_2-k_1} g^{(k_1+k_2-1)}(t)}{(k_1+k_2)!} & \cdots & \frac{t^{k_r-k_1} g^{(k_1+k_r-1)}(t)}{(k_1+k_r)!} \\
\frac{g^{(k_2+k_1-1)}(t)}{(k_2+k_1)!} & \frac{t^{k_2-k_1} g^{(2k_2-1)}(t)}{(2k_2)!} & \cdots & \frac{t^{k_r-k_1} g^{(k_2+k_r-1)}(t)}{(k_2+k_r)!} \\
\vdots & & & \\
\frac{g^{(k_r+k_1-1)}(t)}{(k_r+k_1)!} & \frac{t^{k_2-k_1} g^{(k_r+k_2-1)}(t)}{(k_r+k_2)!} & \cdots & \frac{t^{k_r-k_1} g^{(2k_r-1)}(t)}{(2k_r)!}
\end{vmatrix} \\
&= (t^{2k_1} t^{k_2+k_1}\cdots t^{k_r+k_1}) t^{k_2-k_1}\begin{vmatrix}
\frac{g^{(2k_1-1)}(t)}{(2k_1)!} & \frac{g^{(k_1+k_2-1)}(t)}{(k_1+k_2)!} & \cdots & \frac{t^{k_r-k_1} g^{(k_1+k_r-1)}(t)}{(k_1+k_r)!} \\
\frac{g^{(k_2+k_1-1)}(t)}{(k_2+k_1)!} & \frac{g^{(2k_2-1)}(t)}{(2k_2)!} & \cdots & \frac{t^{k_r-k_1} g^{(k_2+k_r-1)}(t)}{(k_2+k_r)!} \\
\vdots & & & \\
\frac{g^{(k_r+k_1-1)}(t)}{(k_r+k_1)!} & \frac{g^{(k_r+k_2-1)}(t)}{(k_r+k_2)!} & \cdots & \frac{t^{k_r-k_1} g^{(2k_r-1)}(t)}{(2k_r)!}
\end{vmatrix} \\
&= \cdots \\
&= (t^{2k_1} t^{k_2+k_1}\cdots t^{k_r+k_1}) t^{k_2-k_1} \times \cdots \times t^{k_r-k_1}\begin{vmatrix}
\frac{g^{(2k_1-1)}(t)}{(2k_1)!} & \frac{g^{(k_1+k_2-1)}(t)}{(k_1+k_2)!} & \cdots & \frac{g^{(k_1+k_r-1)}(t)}{(k_1+k_r)!} \\
\frac{g^{(k_2+k_1-1)}(t)}{(k_2+k_1)!} & \frac{g^{(2k_2-1)}(t)}{(2k_2)!} & \cdots & \frac{g^{(k_2+k_r-1)}(t)}{(k_2+k_r)!} \\
\vdots & & & \\
\frac{g^{(k_r+k_1-1)}(t)}{(k_r+k_1)!} & \frac{g^{(k_r+k_2-1)}(t)}{(k_r+k_2)!} & \cdots & \frac{g^{(2k_r-1)}(t)}{(2k_r)!}
\end{vmatrix}
\end{aligned}
$$

This implies that $\det((\frac{g^{(k_i+k_j-1)}(t)}{(k_i+k_j)!})) \geq 0$.

Since determinants of all principal submatrices of $(\frac{g^{(i+j-1)}(t)}{(i+j)!})$ is nonnegative, we know that $(\frac{g^{(i+j-1)}(t)}{(i+j)!})$ is positive semidefinite, hence, so is $(\frac{(\int g)^{(i+j)}(t)}{(i+j)!})$.

(2) From the observation of (1) we have

$$
\begin{pmatrix} \frac{1}{2}g'(t) & \frac{1}{6}g^{(2)}(t) \\ \frac{1}{6}g^{(2)}(t) & \frac{1}{24}g^{(3)}(t) \end{pmatrix}
$$

is positive semidefinite for t in $(0,\alpha)$.

Hence $g^{(3)}(t) \geq 0$ for $t \in (0,\alpha)$. From Lemma 3.4 we conclude that g belongs to the class $Q_2(0,\alpha)$. $\square$

Remark 5.2. From Proposition 5.1 if f is k-convex, we have $(\frac{(\int g)^{(i+j)}}{(i+j)!})$ is positive semidefinite. Hence if the local property theorem for the k-convexity is true, then we could conclude that $\int g$ is k-convex in $(0,\alpha)$. $\square$

Remark 5.3. In Propositions 5.1 and [17, Proposition 2.7] we can not drop the condition $f(0) \le 0$. Indeed, consider $h(t) = -\log(t+1) + 1$ for $[0,1)$. Then h is 2-convex from

$$\det(K_2(h;t)) = \frac{1}{72(t+1)^6}.$$

On the contrary, from the similar calculation in Example 5.6.
we have

$$\begin{aligned}
(\frac{h(t)}{t})' &= \frac{-t + (t+1)\log(t+1) - t^2(t+1)}{t^2(t+1)} \\
(\frac{h(t)}{t})^{(2)} &= -\frac{-3t^2 - 2t + 2t^2\log(t+1) + 4\log(t+1) + 2\log(t+1) + 2(t+1)^2}{(t+1)^2t^3} \\
(\frac{h(t)}{t})^{(3)} &= \frac{-11t^3 - 15t^2 - 6t + 6t^3\log(t+1) + 18t^2\log(t+1) + 18t\log(t+1) + 6\log(t+1) - 6(t+1)^3}{(t+1)^3t^4}
\end{aligned}$$

We have then $(\frac{h(t)}{t})^{(3)} < 0$ for some $t \in (0,1)$. Hence $(\frac{h(t)}{t})'$ is not convex, that is, $\frac{h(t)}{t}$ does not belong to $Q_2(0,1)$ by Lemma 3.4. In particular, $\frac{h(t)}{t} \notin P_2(0,1)$. □

Proposition 5.4. Let $\alpha > 0$ and f be a C^2-function on $[0,\alpha)$. Let $f(t) = tg(t)$. Then g is a C^2-function and for any distinct $t_1, t_2, \dots t_n \in (0,\alpha)$ we have

$$[t_1, t_i, t_j]_f = t_1[t_1, t_i, t_j]_g + [t_i, t_j]_g \quad (1 \le i, j \le n).$$

Therefore, if g is n-monotone and n-convex, then f is n-convex.

Proof. For any distinct $t_1, t_2, \dots t_n \in (0,\alpha)$ we have

$$\begin{aligned}
[t_1, t_i, t_j]_f &= \frac{[t_1, t_i]_f - [t_i, t_j]_f}{t_1 - t_j} \\
&= \frac{\frac{f(t_1)-f(t_i)}{t_1-t_i} - \frac{f(t_i)-f(t_j)}{t_i-t_j}}{t_1 - t_j} \\
&= \frac{\frac{t_1g(t_1)-t_ig(t_i)}{t_1-t_i} - \frac{t_ig(t_i)-t_jg(t_j)}{t_i-t_j}}{t_1 - t_j} \\
&= \frac{\frac{t_1(g(t_1)-g(t_i))+(t_1-t_i)g(t_i)}{t_1-t_i} - \frac{(t_i-t_j)g(t_i)-t_j(g(t_j)-g(t_i))}{t_i-t_j}}{t_1 - t_j} \\
&= \frac{\frac{t_1(g(t_1)-g(t_i))}{t_1-t_i} + g(t_i) - g(t_i) + \frac{t_j(g(t_j)-g(t_i))}{t_i-t_j}}{t_1 - t_j} \\
&= \frac{t_1\left\{\frac{g(t_1)-g(t_i)}{t_1-t_i} - \frac{g(t_i)-g(t_j)}{t_i-t_j}\right\} + t_1\frac{g(t_i)-g(t_j)}{t_i-t_j} - t_j\frac{g(t_i)-g(t_j)}{t_i-t_j}}{t_1 - t_j} \\
&= \frac{t_1\left\{\frac{g(t_1)-g(t_i)}{t_1-t_i} - \frac{g(t_i)-g(t_j)}{t_i-t_j}\right\}}{t_1 - t_j} + \frac{(t_1 - t_j)\frac{g(t_i)-g(t_j)}{t_i-t_j}}{t_1 - t_j} \\
&= t_1[t_1, t_i, t_j]_g + [t_i, t_j]_g
\end{aligned}$$

Suppose that g is n-monotone and n-convex. Since g is n-monotone, the correspondent Loewner matrix $([t_i, t_j]_g)$ is positive semidefinite by [15]. Similarily, since g is n-convex, $([t_1, t_i, t_j]_g)$ is positive semidefinite by [15].

Therefore, from the above estimate we have for any distinct $t_1, t_2, \dots, t_n$ in $(0,\alpha)$

$$\begin{aligned}
([t_1, t_i, t_j]_f) &= t_1([t_1, t_i, t_j]_g) + ([t_i, t_j]_g) \\
&\ge 0.
\end{aligned}$$

Hence f is n-convex by [15]. □

Remark 5.5. When f is n-convex with $f(0) \leq 0$, we can give another proof of [17, Theorem 2.2] (that is, $g(t) = \frac{f(t)}{t}$ is $(n-1)$-positive) using Proposition 5.4 as follows:

For any distinct $t_1, t_2, \ldots, t_{n-1}, t_n \in (0, \alpha)$ we have

$$\begin{aligned}
([t_i, t_j]_g)_{1 \leq i,j \leq n-1} &= ([t_n, t_i, t_j]_f)_{1 \leq i,j \leq n-1} - (t_n [t_n, t_i, t_j]_g)_{1 \leq i,j \leq n-1} \\
&= ([t_n, t_i, t_j]_f)_{1 \leq i,j \leq n-1} - (t_n \frac{1}{(t_n - t_j)} \{ \frac{g(t_n) - g(t_i)}{t_n - t_i} - \frac{g(t_i) - g(t_j)}{t_i - t_j} \})_{1 \leq i,j, \leq n-1} \\
&= ([t_n, t_i, t_j]_f)_{1 \leq i,j \leq n-1} - (\frac{1}{(t_n - t_j)} \{ t_n \frac{g(t_n) - g(t_i)}{t_n - t_i} - t_n \frac{g(t_i) - g(t_j)}{t_i - t_j} \})_{1 \leq i,j, \leq n-1} \\
&= ([t_n, t_i, t_j]_f)_{1 \leq i,j \leq n-1} - (\frac{1}{(t_n - t_j)} \{ \frac{f(t_n) - t_n g(t_i)}{t_n - t_i} - t_n \frac{g(t_i) - g(t_j)}{t_i - t_j} \})_{1 \leq i,j \leq n-1} \\
&\geq -(\frac{1}{(t_n - t_j)} \{ \frac{f(t_n) - t_n g(t_i)}{t_n - t_i} - t_n \frac{g(t_i) - g(t_j)}{t_i - t_j} \})_{1 \leq i,j \leq n-1} \ (t \in (0, \alpha))
\end{aligned}$$

Note that since $([t_n, t_i, t_j]_f)_{1 \leq i,j \leq n}$ is positive semidefinite, $([t_n, t_i, t_j]_f)_{1 \leq i,j \leq n-1}$ is positive semidefinite.

When $t_n \to 0$, we get

$$M_{n-1}(g; t) \geq (\frac{-f(0)}{t_i t_j}) \geq 0.$$

Hence g is $(n-1)$-monotone by [15].

In [17, Proposition 2.6] the authors showed that for any 2-convex polynomial f with the degree less than or equal to 5 on $[0, \alpha)$ $g(t) = \frac{f(t)}{t}$ is 2-monotone.

We have another affirmative example for the implication from (1) to (3).

Example 5.6. Let $f(t) = -\log(t+1)$ for $[0, 1)$. Then f is 2-convex. Indeed, since for any $n \in \mathbf{N}$ $f^{(n)}(t) = (-1)^n \frac{(n-1)!}{(t+1)^n}$, we have

$$\begin{aligned}
K_2(f, t) &= (\frac{f^{(i+j)}(t)}{(i+j)!}) \\
\det(K_2(f, t)) &= \frac{1}{72(t+1)^6}.
\end{aligned}$$

Since $f^{(2)}(t) \geq 0$, $f^{(4)}(t) \geq 0$, and $\det(K_2(f, t)) \geq 0$ for $t \in [0, 1)$, $K_2(f, t)$ is positive semidefinite for $t \in [0, 1)$. Therefore, f is 2-convex by [10].

Let $g(t) = \frac{f(t)}{t} = -\frac{\log(t+1)}{t}$. Then we will show that g is 2-monotone on $(0, 1)$. We have, then,

$$\begin{aligned}
g'(t) &= \frac{-t + (t+1)\log(t+1)}{t^2(t+1)} \\
g^{(2)}(t) &= -\frac{-3t^2 - 2t + 2t^2 \log(t+1) + 4t \log(t+1) + 2\log(t+1)}{(t+1)^2 t^3} \\
g^{(3)}(t) &= \frac{-11t^3 - 15t^2 - 6t + 6t^3 \log(t+1) + 18t^2 \log(t+1) + 18t \log(t+1) + 6\log(t+1)}{(t+1)^3 t^4} \\
\det(M_2(g, t)) &= 12 \frac{-5t^2 - 6t + 2t^2 \log(t+1) + 8t \log(t+1) + 6\log(t+1)}{t^4(t+1)^4}.
\end{aligned}$$

From the simple calculations we conclude that all $g', g^{(3)}, \det(M_2(g, t))$ are nonnegative. Note that $(t+1)^3 t^4 g^{(3)}(t) = \frac{1}{10}t^6 - \frac{3}{10}t^5 + \frac{3}{2}t^4 + O(t^7)$ and $t^4(t+1)^4 \det(M_2(g, t)) = \frac{1}{6}t^4 - \frac{2}{15}t^5 + \frac{1}{10}t^6 + O(t^7)$ by using the Maclaurin series of $\log(t+1)$.

Hence $M_2(g, t)$ is positive semidefinite. Therefore, g is 2-monotone.

Generally, however, even if $n = 2$, the assertion (1) does not necesarily imply the assertion (3).

We need the following simple observation.

Lemma 5.7. For $\alpha, \beta > 0$ we define a function $h\colon [0,\alpha) \to [0,\beta)$ by $h(t) = \frac{\beta}{\alpha}t$. Then

(1) If $f \in K_n([0,\alpha))$, then $f \circ h^{-1} \in K_n([0,\beta))$.

(2) If $\frac{f(t)}{t} \in M_n((0,\alpha))$, then $\frac{(f\circ h^{-1})(t)}{t} \in M_n((0,\beta))$.

Proof. (1): Let $A, B \in M_n(\mathbf{C})$ with $0 \le A, B \le \beta I_n$, where I_n is the identity matrix in $M_n(\mathbf{C})$. Take for any $\lambda \in [0,1]$. Since $f \in K_n([0,\alpha))$ and $0 \le \frac{\alpha}{\beta}A, \frac{\alpha}{\beta}B \le \alpha I_n$, we have

$$\begin{aligned} f(\lambda\frac{\alpha}{\beta}A + (1-\lambda)\frac{\alpha}{\beta}B) &\le \lambda f(\frac{\alpha}{\beta}A) + (1-\lambda)f(\frac{\alpha}{\beta}B) \\ f(\frac{\alpha}{\beta}(\lambda A + (1-\lambda)B) &\le \lambda f(\frac{\alpha}{\beta}A) + (1-\lambda)f(\frac{\alpha}{\beta}B) \\ (f\circ h^{-1})(\lambda A + (1-\lambda)B) &\le \lambda (f\circ h^{-1})(A) + (1-\lambda)(f\circ h^{-1})(B) \end{aligned}$$

Hence $f \circ h^{-1} \in K_n([0,\beta))$.

(2): Let $A, B \in M_n(\mathbf{C})$ with $0 \le A \le B \le \beta I_n$. Since $\frac{f(t)}{t} \in M_n((0,\alpha))$ and $0 \le \frac{\alpha}{\beta}A \le \frac{\alpha}{\beta}B \le \alpha I_n$, we have

$$\begin{aligned} 0 &\le (\frac{\alpha}{\beta}A)^{-1}f(\frac{\alpha}{\beta}A) \le (\frac{\alpha}{\beta}B)^{-1}f(\frac{\alpha}{\beta}B) \\ 0 &\le \frac{\beta}{\alpha}A^{-1}f(\frac{\alpha}{\beta}A) \le \frac{\beta}{\alpha}B^{-1}f(\frac{\alpha}{\beta}B) \\ 0 &\le \frac{\beta}{\alpha}A^{-1}(f\circ h^{-1})(A) \le \frac{\beta}{\alpha}B^{-1}(f\circ h^{-1})(B) \\ 0 &\le A^{-1}(f\circ h^{-1})(A) \le B^{-1}(f\circ h^{-1})(B). \end{aligned}$$

Hence $\frac{(f\circ h^{-1})(t)}{t} \in M_n((0,\beta))$. □

Theorem 5.8. For any $\alpha > 0$ there is a 2-convex function f on $[0,\alpha)$, but $g(t) = \frac{f(t)}{t}$ is not 2-monotone on $(0,\alpha)$.

Proof. Let $f(t) = t + \frac{1}{2}t^2 + \frac{1}{3}t^3 - \log(t+1)$ and $g(t) = \frac{f(t)}{t} = 1 + \frac{1}{2}t + \frac{1}{3}t^2 - \frac{\log(t+1)}{t}$.

Since

$$\begin{aligned} K_2(f,t) &= \begin{pmatrix} \frac{1}{2} + t + \frac{1}{2(t+1)^2} & \frac{1}{3} - \frac{1}{3(t+1)^3} \\ \frac{1}{3} - \frac{1}{3(t+1)^3} & \frac{1}{4(t+1)^4} \end{pmatrix} \\ \det(K_2(f,t)) &= -\frac{1}{72}\frac{1}{(t+1)^6}(27t^2 - 36t - 18 + 126t^3 + 8t^6 + 48t^5 + 120t^4), \end{aligned}$$

$K_2(f,t)$ is positive definite for some $[0,\beta)$ $(\alpha > 0)$. For example $\beta = 0.1$.

On the contrary,

$$M_2(g,t) = \begin{pmatrix} \frac{1}{2} + \frac{2}{3}t - \frac{1}{(t+1)t} + \frac{\log(t+1)}{t^2} & \frac{1}{3} + \frac{1}{2(t+1)^2t} + \frac{1}{(t+1)t^2} - \frac{\log(t+1)}{t^3} \\ \frac{1}{3} + \frac{1}{2(t+1)^2t} + \frac{1}{(t+1)t^2} - \frac{\log(t+1)}{t^3} & -\frac{1}{3(t+1)^3t} - \frac{1}{2(t+1)^2t^2} - \frac{1}{(t+1)t^3} + \frac{\log(t+1)}{t^4} \end{pmatrix}$$

$$\begin{aligned} \det(M_2(g,t)) = \frac{1}{36}\frac{1}{t^4(t+1)^4}\{&-4t^8 - 16t^7 - 24t^6 - 96t^5 - 237t^4 - 246t^3 - 126t^2 - 36t \\ &+ 48t^5\log(t+1) + 210t^4\log(t+1) + 360t^3\log(t+1) + 306t^2\log(t+1) + 144t\log(t+1) + 36\log(t+1)\} \end{aligned}$$

Since $t > \log(t+1)$, there must exist a sufficiently small $t_0 \in (0,\beta)$ such that $\det(M_2(g,t_0)) < 0$. For example $\det(M_2(g,t))(1.0 \times 10^{-9}) = -2.7777778682 \times 10^{17}$ using Maple.

Therefore, f is 2-convex on $[0,\beta)$, but g is not 2-monotone on $(0,\beta)$.

Let $h\colon [0,\alpha) \to [0,\beta)$ by $h(t) = \frac{\beta}{\alpha}t$ and set $F = f \circ h$. Then $F \in K_2([0,\alpha)$. Suppose that $\frac{F(t)}{t}$ is 2-monotone in $(0,\alpha)$. Then by Lemma 5.7(2) $\frac{f(t)}{t}(= g(t))$ is 2-monotone on $(0,\beta)$. This is a contradiction. Hence $\frac{F(t)}{t} \notin M_2((0,\alpha))$. $\square$

Next we give an affirmative answer for the convex implication from $(3)_n$ to $(1)_n$.

Lemma 5.9. Let f be a C^2 function on some interval $[0,\alpha)$. Let take $t \in [0,\alpha)$ and consider a function $[t,z]_f$. Then

$$\frac{d}{dz}[t,z]_f = [t,z,z]_f.$$

Proof. Direct computation. $\square$

Theorem 5.10. Let $f \in \mathbf{Q}_n([0,\alpha))$. Suppose that $\frac{f(t)-f(0)}{t}$ is $(n-1)$-monotone on $(0,\alpha)$. Then f is $(n-1)$-convex. In particular, if $f(0) = 0$, then $(n-1)$-monotonicity of $\frac{f(t)}{t}$ implies $(n-1)$-convexity of f.

Proof. Since $f \in Q_n([0,\alpha))$, for any $z \in [0,\alpha)$ $h(t) = [t,z,z]$ is (n - 1)-monotone [5, XV Lemma 4]. Hence for any $\lambda_1,\ldots,\lambda_{n-1} \in [0,\alpha)$ $([\lambda_i,\lambda_j]_h) \geq 0$, that is, for any $\xi = (\xi_1,\xi_2,\ldots,\xi_{n-1}) \in \mathbf{C}^{n-1}$

$$(([\lambda_i,\lambda_j]_h)\xi \mid \xi) \geq 0,$$

$$\sum_{i,j=1}^{n-1}[\lambda_i,\lambda_j]_h\xi_j\overline{\xi_i} \geq 0.$$

Since

$$\begin{aligned}\sum_{i,j=1}^{n-1}[\lambda_i,\lambda_j]_h\xi_j\overline{\xi_i} &= \sum_{i,j=1}^{n-1}\frac{1}{\lambda_i-\lambda_j}(h(\lambda_i)-h(\lambda_j))\xi_j\overline{\xi_i}\\ &= \sum_{i,j=1}^{n-1}\frac{1}{\lambda_i-\lambda_j}([\lambda_i,z,z]_f-[\lambda_j,z,z]_f\xi_j\overline{\xi_i}\\ &= \sum_{i,j=1}^{n-1}\frac{1}{\lambda_i-\lambda_j}(\frac{d}{dz}[\lambda_i,z]_f-\frac{d}{dz}[\lambda_j,z]_f)\xi_j\overline{\xi_i} \quad \text{(Lemma 5.9)},\end{aligned}$$

we have then

$$\begin{aligned}0 &\leq \int_0^z(\sum_{i,j=1}^{n-1}\frac{1}{\lambda_i-\lambda_j}(\frac{d}{dz}[\lambda_i,z]_f-\frac{d}{dz}[\lambda_j,z]_f)\xi_j\overline{\xi_i})dz\\ &= \sum_{i,j=1}^{n-1}\frac{1}{\lambda_i-\lambda_j}\{([\lambda_i,z]_f-[\lambda_i,0]_f)-([\lambda_j,z]_f-[\lambda_j,0]_f)\}\xi_j\overline{\xi_i}\\ &= \sum_{i,j=1}^{n-1}\frac{1}{\lambda_i-\lambda_j}\{([\lambda_i,z]_f-[\lambda_j,z]_f\}\xi_j\overline{\xi_i})-([\lambda_i,0]_f-[\lambda_j,0]_f\}\xi_j\overline{\xi_i})\}\\ &= \sum_{i,j=1}^{n-1}[\lambda_i,\lambda_j,z]_f\xi_j\overline{\xi_i}-\sum_{i,j=1}^{n-1}[\lambda_i,\lambda_j,0]_f\xi_j\overline{\xi_i}.\end{aligned}$$

Hence, $([\lambda_i,\lambda_j,0]_f]) \leq ([\lambda_i,\lambda_j,z]_f)$. Note that $([\lambda_i,\lambda_j,0]_f]) = ([\lambda_i,\lambda_j]_{\frac{f(t)-f0)}{t}})$.

Since $\frac{f(t)-f(0)}{t}$ is $(n-1)$-monotone by the assumption, $([\lambda_i,\lambda_j]_{\frac{f(t)-f0)}{t}})$ is positive semidefinite, and so is $([\lambda_i,\lambda_j,0]_f])$. Hence $([\lambda_i,\lambda_j,z]_f)$ is positive semidefinite. Since $z \in [0,\alpha)$ is an arbitrary, $([\lambda_i,\lambda_j,\lambda_{n-1}]_f)$ is positive semidefinite, that is, f is $(n-1)$-convex by [14]. $\square$

Remark 5.11. From [17, Lemma 2.3(2)] if $f(0) \le 0$, the n-monotonicity of $\frac{f(t)-f(0)}{t}$ implies the n-monotonicity of $\frac{f(t)}{t}$. But it is not obvious that the converse is true.

Suppose that $\frac{f(t)}{t}$ is operator monotone. We have then

$$\begin{aligned} f \text{ is operator convex} &\Rightarrow f(t) - f(0) \text{ is operator convex} \\ &\Rightarrow \frac{f(t)-f(0)}{t} \end{aligned}$$

is operator monotone by [8].

Using the same idea we could conclude that the n-monotonicity and n-convexity of $\frac{f(t)}{t}$ implies $(n-1)$-monotonicity of $\frac{f(t)-f(0)}{t}$ by Proposition 5.4 and [17, Theorem 2.2]. Indeed, if $\frac{f(t)}{t}$ is n-monotone and n-convex, then f is n-convex by Proposition 5.4. Hence $f(t) - f(0)$ is n-convex. Therefore, $\frac{f(t)-f(0)}{t}$ is $(n-1)$-monotone by [17, Theorem 2.2].

We obtain the following characterization of n-monotonicity of $\frac{f(t)-f(0)}{t}$ from the n-monotonicity of the first derivative of f.

Theorem 5.12. Let $f\colon [0,\alpha) \to \mathbf{R}$ be a C^1 function. Suppose that f' is n-monotone. Then $\frac{f(t)-f(0)}{t}$ is n-monotone on $(0,\alpha)$.

Proof. Note that if $g\colon [0,\alpha) \to \mathbf{R}$ is n-monotone, then for any $u \in [0,1]$ $h(t) = g(ut)$ is n-monotone on $[0,\alpha)$.

Since f' is n-monotone, for any $\lambda_1, \lambda_2, \dots, \lambda_n \in (0,\alpha)$ $([\lambda_i,\lambda_j]_{f'})$ is positive semidefinite, that is, for any $\xi_1, \xi_2, \dots, \xi_n \in \mathbf{C}$

$$\sum_{i,j=1}^n \frac{1}{\lambda_i - \lambda_j}\{f'(\lambda_i) - f'(\lambda_j)\}\xi_j\overline{\xi_i} \ge 0.$$

Hence for any $u \in [0,1]$

$$\sum_{i,j=1}^n \frac{1}{\lambda_i - \lambda_j}\{f'(u\lambda_i) - f'(u\lambda_j)\}\xi_j\overline{\xi_i} \ge 0.$$

We have then for any $u \in [0,1]$

$$\begin{aligned} 0 &\le \int_0^1 \sum_{i,j=1}^n \frac{1}{\lambda_i - \lambda_j}\{f'(u\lambda_i) - f'(u\lambda_j)\}\xi_j\overline{\xi_i}du \\ &= \sum_{i,j=1}^n \frac{1}{\lambda_i - \lambda_j}\int_0^1 \left\{\frac{1}{\lambda_i}f(u\lambda_i) - \frac{1}{\lambda_j}f(u\lambda_j)\right\}' du\xi_j\overline{\xi_i} \\ &= \sum_{i,j=1}^n \frac{1}{\lambda_i - \lambda_j}\left\{\frac{f(\lambda_i)-f(0)}{\lambda_i} - \frac{f(\lambda_j)-f(0)}{\lambda_j}\right\}\xi_j\overline{\xi_i} \end{aligned}$$

Therefore, $[[\lambda_i,\lambda_j]_{\frac{f(t)-f(0)}{t}}]$ is positive semidefinite, and $\frac{f(t)-f(0)}{t}$ is n-monotone. $\square$

We have, then, the following relation between the class $Q_n([0,\alpha))$ and the class $K_{n-1}([0,\alpha))$.

Corollary 5.13. Let $f \in Q_n([0,\alpha))$. Suppose that f' is $(n-1)$-monotone. Then f is $(n-1)$-convex.

Proof. From Theorem 5.12 $\frac{f(t)-f(0)}{t}$ is $(n-1)$-monotone on $(0,\alpha)$. Hence f is $(n-1)$-convex by Theorem 5.10. $\square$

Remark 5.14. It is not true that $Q_n(I) \subset K_{n-1}(I)$ for an interval in $[0,\infty)$. Indeed, If $0 < \alpha < 1$, t^α is n-monotone, but not 2-convex. Hence if $n \ge 3$ and $\beta > 0$, then $t^\alpha \in Q_n([0,\beta))$, but $t^\alpha \notin K_{n-1}([0,\beta))$.

As pointed out in Proposition 3.5 in [17] the implication from (3) to (1) is not true even if $n = 1$. We then have another observation as in Theorem 5.8 when $n = 2$ using Maple.

Theorem 5.15. For any $\alpha > 0$ there is a 2-monotone function g on $(0, \alpha)$, but $f(t) = tg(t)$ is not 2-convex on $[0, \alpha)$.

Proof. Let f be a 2-convex function defined by $f(t) = t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{5}t^5$. Then we know that $K_2(f, t) = \frac{1}{72} + \frac{1}{12}t - \frac{23}{24}t^2 - 2t^3 - 2t^4$ is positive definite on $[0, 0.14)$, but negative definite on $[0.15, 1)$.

On the contrary, $g(t) = \frac{f(t)}{t} = 1 + \frac{1}{2}t + \frac{1}{3}t^2 + \frac{1}{4}t^4 + \frac{1}{5}t^4$ is 2-monotone on $[0, 0.17]$ from which $M_2(g, t) = \frac{1}{72} + \frac{1}{15}t - \frac{77}{120}t^2 - t^3 - \frac{4}{5}t^4$ is positive definite on $[0, 0.17]$.

Hence $\frac{f(t)}{t} \in M_2([0, 0.17])$, but $f \notin K_2([0, 0.17])$.

Let $\alpha > 0$ and $\beta = 0.17$, and define $h \colon [0, \alpha) \to [0, \beta)$ by $h(t) = \frac{\beta}{\alpha}t$. Define $G(t) = \frac{(f \circ h)(t)}{t}$. Then $G \in M_2([0, \alpha))$ by Lemma 5.7(2). As in the proof of Theorem 5.10, however, $tG(t) \notin K_2([0, \alpha))$. Indeed, suppose that $tG(t) \in K_2([0, \alpha))$. Then $(f \circ h) \in K_2([0, \alpha))$. By Lemma 5.7(1) we know that $f \in K_2([0, \beta))$, and a contradiction. Hence $tG(t) \notin K_2([0, \alpha))$. $\square$

Before closing this note we summarize several observations and a problem between condition (1) and condition (3), which are presented at the first part in this section.

Theorem 5.16. Let $0 < \alpha \leq \infty$ and f be a real valued function in $[0, \alpha)$ with $f(0) \leq 0$. Let $g(t) = \frac{f(t)}{t}$. Suppose that f is a C^2-function.

(i) If g is n-monotone and n-convex, then f is n-convex.
(ii) If f is 2-convex, then $g \in Q_2(0, \alpha)$, but it does not necessarily imply that g is 2-monotone.
(iii) If f in $Q_{n+1}([0, \alpha))$ with $f(0) = 0$ and g is n-monotone, then f is n-convex. In particular, if f is $(n + 1)$-monotone, then the implication from $(3)_n$ to $(1)_n$ holds.

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Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan
E-mail address: osaka@se.ritsumei.ac.jp

Prof. Emeritus of Tokyo Metropolitan University, 201 11-10 Nakane 1-chome, Meguro-ku, Tokyo, Japan
E-mail address: juntomi@med.email.ne.jp